# ON AN AXISYMMETRICAL PROBLEM OF THE THEORY OF ELASTICITY FOR A HOLLOW CYLINDER

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This paper considers a particular case of the mixed axisymmetrical problem of the theory of elasticity. It concerns the state of stress which occurs when an absolutely rigid semi-infinite cylinder is pressed into a thick-walled tube (see fig.).

It is required to determine the stress function $\chi(r, z)$ which satisfies the biharmonic equation in the cylindrical system of coordinates

$$\nabla^4\chi = 0 \tag{1}$$

and the boundary conditions on the side surfaces of the tube

$$\sigma_r = \frac{\partial}{\partial z}\left(\nu\nabla^2\chi - \frac{\partial^2\chi}{\partial r^2}\right) = 0 \quad \text{for} \begin{cases} r = r_2, & -\infty < z < +\infty \\ r = r_1, & 0 < z < +\infty \end{cases} \tag{2}$$

$$\tau_{rz} = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\chi - \frac{\partial^2\chi}{\partial z^2}\right] = 0 \quad \text{for} \begin{cases} r = r_2, & -\infty < z < +\infty \\ r = r_1, & -\infty < z < +\infty \end{cases} \tag{3}$$

$$u = -\frac{1+\nu}{E}\frac{\partial^2\chi}{\partial r\partial z} = u_0 \quad \text{for} \quad r = r_1, \ -\infty < z < 0 \tag{4}$$

The method of solving the problem formulated was suggested by Danilevskii and Al'perin [1], and was subsequently used in paper [2].

We construct an auxiliary solution of equation (1)

$$\chi_0(r, z, m) = e^{mz}\varphi(r)$$

where $m$ is a complex parameter. In accordance with [2]

$$\varphi(r) = AJ_0(mr) + BmrJ_1(mr) + CY_0(mr) + DmrY_1(mr) \tag{5}$$

The relationships between $A(m)$, $B(m)$, $C(m)$ and $D(m)$ are determined in such a way that this solution satisfies the first boundary condition (2) and boundary conditions (3):

$$A\,[n\eta J_0(n\eta) - J_1(n\eta)] + Bn\eta\,[(2\nu - 1)\,J_0(n\eta) + n\eta J_1(n\eta)] + \\ + C\,[n\eta Y_0(n\eta) - Y_1(n\eta)] + Dn\eta\,[(2\nu - 1)\,Y_0(n\eta) + n\eta Y_1(n\eta)] = 0 \qquad (6)$$

$$AJ_1(n\eta) - B\,[n\eta J_0(n\eta) + 2\,(1-\nu)\,J_1(n\eta) + CY_1(n\eta)] - \\ - D\,[n\eta Y_0(n\eta) + 2\,(1-\nu)\,Y_1(n\eta)] = 0 \qquad (7)$$

$$AJ_1(\eta) - B\,[\eta J_0(\eta) + 2\,(1-\nu)\,J_1(\eta)] + CY_1(\eta) - D\,[\eta Y_0(\eta) + 2\,(1-\nu)\,Y_1(\eta)] = 0 \qquad (8)$$

On the basis of (5), (6), (7), (8) we obtain

$$\chi_\rho(\lambda,\ \rho,\ \eta) = \frac{r_1^3 e^{\lambda\eta}\,\Delta_\chi(\rho,\ \eta)}{\eta^2\,\Delta_r(\eta)}\,k(\eta) \qquad (9)$$

where

$$z = \lambda r_1, \qquad r = \rho r_1, \qquad mr_1 = \eta, \qquad mr_2 = n\eta, \qquad n = \frac{r_2}{r_1}$$

$$\Delta_\chi(\rho,\ \eta) = \begin{vmatrix} J_0(\rho\eta) & Y_0(\rho\eta) & \rho\eta Y_1(\rho\eta) & -\rho\eta J_1(\rho\eta) \\ J_1(\eta) & Y_1(\eta) & -\Lambda_2[Y(\eta)] & \Lambda_2[J(\eta)] \\ J_1(n\eta) & Y_1(n\eta) & -\Lambda_2[Y(n\eta)] & \Lambda_2[J(n\eta)] \\ \Lambda_1[J(n\eta)] & \Lambda_1[Y(n\eta)] & \Lambda_3[Y(n,\eta)] & -\Lambda_3[J(n\eta)] \end{vmatrix} \qquad (10)$$

$$\Lambda_1[J(u)] = J_0(u)\,u - J_1(u), \qquad \Lambda_2[J(u)] = 2\,(1-\nu)\,J_1(u) + uJ_0(u)$$

$$\Lambda_3[J(u)] = u\,[(2\nu - 1)\,J_0(u) + uJ_1(u)]$$

$$\Delta_r(\rho,\ \eta) = \begin{vmatrix} -\Lambda_1[J(\rho\eta)] & -\Lambda_1[Y(\rho\eta)] & -\Lambda_3[Y(\rho\eta)] & \Lambda_3[J(\rho\eta)] \\ J_1(\eta) & Y_1(\eta) & -\Lambda_2[Y(\eta)] & \Lambda_2[J(\eta)] \\ J_1(n\eta) & Y_1(n\eta) & -\Lambda_2[Y(n\eta)] & \Lambda_2[J(n\eta)] \\ \Lambda_1[J(n\eta)] & \Lambda_1[Y(n\eta)] & \Lambda_3[Y(n\eta)] & -\Lambda_3[J(n\eta)] \end{vmatrix} \qquad (11)$$

$$\Delta_r(\eta) = \Delta_r(1,\ \eta), \qquad k(\eta) = -\frac{B\eta^2\,\Delta_r(\eta)}{r_1^3\Delta} \qquad (12)$$

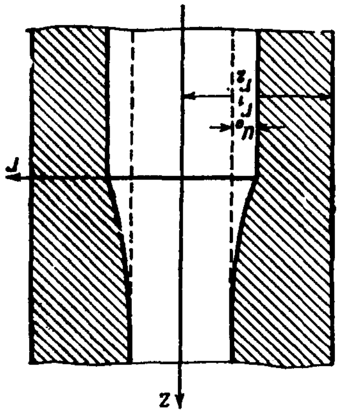
$\Delta$ is the minor corresponding to the element of the first line of the fourth column in the determinant. The group of elements of determinants (10) and (11) contain Bessel functions of the second kind, which have a logarithmic singularity at the point $\eta = 0$; the determinants themselves, however, will be unique functions.

This follows from the fact that $Yn(\eta)$ receives an increment $4iJ_n(\eta)$ in passing through the origin of coordinates, and the determinants considered receive an increment which may be represented in the form of a sum of determinants with equal columns.

The uniqueness of the analogous determinants encountered in the following expressions is established in the same manner. Considering $\eta$ as a parameter, we form an integral

$$\chi(\rho, \lambda) = \int_{-i\infty}^{0-, +i\infty} \chi_0(\lambda, \rho, \eta)\, d\eta \tag{13}$$

which will be the solution of equation (1), if it, together with its derivatives with respect to $\rho$ and $\lambda$ up to the fourth order inclusive, converges absolutely and uniformly in the region $1 < \rho < n$, $|\lambda| < \infty$. This solution satisfies the first boundary condition (2) and boundary conditions (3).




It remains to determine the function $k(\eta)$ in such a fashion that the second boundary condition (2) and boundary condition (4) are satisfied. From (2) and (9) it follows

$$\sigma_r = -\frac{1}{r_1} \int_{-i\infty}^{0-, +i\infty} k(\eta)\, e^{\lambda\eta}\, d\eta \qquad \text{for } \rho = 1,\ |\lambda| < \infty \tag{14}$$

$$u = \frac{1+\nu}{E} \int_{-i\infty}^{0-, +i\infty} \psi(\eta)\, e^{\lambda\eta}\, d\eta \qquad \text{for } \rho = 1,\ |\lambda| < \infty \tag{15}$$

$$\Delta_u(\rho, \eta) = \begin{vmatrix} -J_1(\rho\eta) & -Y_1(\rho\eta) & \rho\eta Y_0(\rho\eta) & -\rho\eta J_1(\rho\eta) \\ J_1(\eta) & Y_1(\eta) & -\Lambda_2[Y(\eta)] & \Lambda_2[J(\eta)] \\ J_1(n\eta) & Y_1(n\eta) & -\Lambda_2[Y(n\eta)] & \Lambda_2[J(n\eta)] \\ \Lambda_1[J(n\eta)] & \Lambda_1[Y(n\eta)] & \Lambda_3[Y(n\eta)] & -\Lambda_3[J(n\eta)] \end{vmatrix} \tag{16}$$

$$\Delta_u(\eta) = \Delta_u(1, \eta)$$

The boundary condition (14) becomes the second boundary condition (2) if $k(\eta)$ is regular in the region $\mathrm{Re}(\eta) \leqslant 0$, $\eta \neq 0$ and the requirements of Jordan's lemma in this region are satisfied. The boundary condition

(15) becomes boundary condition (4) if

$$\psi(\eta) = k(\eta)\frac{\Delta_u(\eta)}{\Delta_r(\eta)} \tag{17}$$

is regular in the region Re$(\eta) \geqslant 0$, $\eta \neq 0$ and the requirements of Jordan's lemma are satisfied in this region. At the origin of coordinates $\Psi(\eta)$ must have a simple pole with the residue

$$\operatorname{res}[\psi(\eta)]|_{\eta=0} = -\frac{Eu_0}{2\pi i(1+\nu)}, \qquad \operatorname{res}[k(\eta)]|_{\eta=0} = -\frac{Eu_0(n^2-1)}{2\pi i[1-\nu+n^2(1+\nu)]}$$

To construct the function $k(\eta)$ following Al'perin [1], we form the infinite product

$$\Pi(\eta) = \prod_{k=1}^{\infty}\frac{(1-\eta/a_k)(1-\eta/\bar{a}_k)}{(1-\eta/b_k)(1-\eta/\bar{b}_k)} \tag{18}$$

where $a_k$ and $a_k$ are the roots of the equation $\Delta_r(\eta) = 0$, situated on the right-hand half-plane, and $b_k$ and $b_k$ are roots of the equation $\Delta_u(\eta) = 0$ situated on the right-hand half-plane.

Investigating $\Pi(\eta)$ at infinity by the method applied by Al'perin [1] and again in paper [2], we obtain

$$\Pi(\eta) \approx [1+0(1)]\sqrt{-\eta\frac{1-\nu+n^2(1+\nu)}{2(1-\nu^2)(n^2-1)}} \tag{19}$$

Now it is easy to establish that

$$k(\eta) = -\frac{Eu_0(n^2-1)}{2\pi i[1-\nu+n^2(1+\nu)]}\frac{\Pi(\eta)}{\eta} \tag{20}$$

satisfies all the requirements enumerated above, and the function

$$\chi(\rho,\lambda) = \frac{Eu_0(n^2-1)r_1^2}{2\pi i[1-\nu+n^2(1+\nu)]}\int\frac{\Pi(\eta)}{\eta^3}\frac{\Delta_\chi(\rho,\eta)}{\Delta_r(\eta)}e^{\lambda\eta}\,d\eta \tag{21}$$

is a solution of the boundary-value problem considered. The limiting values of the components of the stress tensor as $\lambda \to \pm\infty$ are found analogously [2]:

$$\lim_{\lambda\to+\infty}\sigma_r = 0, \qquad \lim_{\lambda\to-\infty}\sigma_r = \frac{Eu_0}{r_1[1-\nu+n^2(1+\nu)]}\frac{\rho^2-n^2}{\rho^2}, \qquad \lim_{\lambda\to\pm\infty}\sigma_z = 0$$

$$\lim_{\lambda\to+\infty}\sigma_\theta = 0, \qquad \lim_{\lambda\to-\infty}\sigma_\theta = \frac{Eu_0}{r_1[1-\nu+n^2(1+\nu)]}\frac{\rho^2+n^2}{\rho^2}, \qquad \lim_{\lambda\to\pm\infty}\sigma_{rz} = 0$$

$$\lim_{\lambda\to+\infty}u = 0, \qquad \lim_{\lambda\to-\infty}u = \frac{u_0[(1+\nu)n^2+(1-\nu)\rho^2]}{\rho[1-\nu+n^2(1+\nu)]}$$

From these expressions it may be seen that the state of stress in the thick-walled tube as $\lambda \to -\infty$ becomes plane, corresponding to Lamé's problem with the boundary conditions $u = u_0$ for $r = r_1$ and $\sigma_2 = 0$ for $r = r_2$.

In conclusion, we determine the concentration of radial stress and the character of discontinuity of the side surface of the tube $\rho = 1$ as $\lambda \to 0$.

Let us consider expression (14) for $\sigma_r$ as $\rho = 1$

$$\sigma_r = \frac{Eu_0 (n^2 - 1)}{2\pi i r_1 [1 - \nu + n^2 (1 + \nu)]} \int_{-i\infty}^{0-, +i\infty} \frac{\Pi(\eta)}{\eta} e^{\lambda\eta} d\eta$$

We put $|\lambda| \eta = v$, $\lambda < 0$

$$\sigma_r = \frac{Eu_0 (n^2 - 1)}{2\pi i [1 - \nu + n^2 (1 + \nu)]} \int_{-i\infty}^{0-, +i\infty} \Pi\left(\frac{v}{|\lambda|}\right) \frac{e^{-v}}{v} dv =$$

$$= \frac{Eu_0 (n^2 - 1)}{2\pi i [1 - \nu + n^2 (1 + \nu)]} \int_C \Pi\left(\frac{v}{|\lambda|}\right) \frac{e^{-v}}{v} dv$$

The contour of integration $C$ consists of the imaginary axis with the symmetrically excluded portion of length $2a$ replaced by a semi-circular arc of radius $a$, situated in the region $\mathrm{Re}\, v < 0$.

Since $|v| \geqslant a$ everywhere on $C$, the ratio $|v| / |\lambda|$ may be made as large as desired for a sufficiently small $\lambda$. Using the asymptotic representation of $\Pi(\eta)$ (19), we find the expression for $\sigma_r$ suitable at $\rho = 1$ for small $\lambda < 0$:

$$\sigma_r = -\frac{Eu_0 (n^2 - 1)}{2\pi i r_1 \sqrt{2 |\lambda| (1 - \nu^2)(n^2 - 1) [1 - \nu + n^2 (1 + \nu)]}} \int_C \frac{e^{-v}}{\sqrt{-v}} dv$$

In paper [2] it was shown that

$$\int_C \frac{e^{-v}}{\sqrt{-v}} dv = 2i\sqrt{\pi}$$

Consequently,

$$\sigma_r \approx -\frac{Eu_0 (n^2 - 1)}{r_1 \sqrt{2\pi |\lambda| (1 - \nu^2)(n^2 - 1) [1 - \nu + n^2 (1 + \nu)]}} \tag{22}$$

In an analogous manner we find the expression for $u$, suitable for small $\lambda > 0$ and $\rho = 1$:

$$u = u_0 - 2u_0 \sqrt{\frac{2\lambda (1 - \nu^2)(n^2 - 1)}{\pi [1 - \nu + n^2 (1 + \nu)]}} \tag{23}$$

The solution for the exterior of the cylinder is obtained by a limiting process as $n \to \infty$.

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